Strong Coupling Quantum Gravity and Physics beyond the Planck Scale

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Abstract

We propose a renormalization prescription for the Wheeler-DeWitt equation of (3+1)-dimensional Einstein gravity and also propose a strong coupling expansion as an approximation scheme to probe quantum geometry at length scales much smaller than the Planck length. We solve the Wheeler-DeWitt equation to the second order in the expansion in a class of local solutions and discuss problems arising in our approach.

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I INTRODUCTION

Nonperturbative effects of quantum gravity would play a vital role in the physics at the Planck scale and drastically change the concept of spacetime below the Planck length. The perturbative non-renormalizability of quantum gravity[1] is probably a sign that nonperturbative approaches to quantum gravity are essential to study spacetime structure at short distances below the Planck length.

Although we have not yet understood the physics at and beyond the Planck scale, there have been frequent suggestions that the concept of spacetime loses its meaning below the Planck length. In fact, many different approaches to quantum gravity have led to the conclusion of the existence of a minimum length[3]. The concept of the minimum length is very suggestive. Since it may imply discrete nature of spacetime in quantum gravity[3], the number of dynamical degrees of freedom will be much smaller than what one naively expects. Witten has proposed an interesting idea that the physics at and beyond the Planck scale is described by a topological quantum field theory with a finite dimensional Hilbert space[4]. It has also been argued by 't Hooft that the finiteness of entropy and information in a black hole is evidence for the discreteness of spacetime and that the number of degrees of freedom is given by the area of the event horizon in Planck units. This has led 't Hooft[5] and Susskind[6] to make the holographic hypothesis that physical states are described by a quantum field theory on the surface of the black hole, and a realization of the idea has been discussed by Smolin in topological quantum field theory[7].

A canonical quantization of gravity is one of basic approaches to quantum gravity. A major advance has been developed in the canonical quantum gravity proposed by Ashtekar[8]. In terms of Ashtekar's new variables, a large class of solutions to the Hamiltonian constraint has been constructed in the loop representation[9]. A remarkable observation is that the area and the volume operators have discrete spectrum, i.e., they are quantized in Planck units[10, 11].

In this paper, we will take another approach to the canonical quantum gravity based on the Wheeler-DeWitt equation[12] and investigate spacetime structure beyond the Planck scale. Main technical obstacles to prevent the study of the short distance physics are that WKB approximation will not be applicable at short dis-

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[3]For a recent review, see Ref.[2].

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tances and that the Wheeler-DeWitt equation is ill-defined without regularization. In this paper, we propose an approximation scheme which will be well-suited to probe quantum geometry at length scales much smaller than the Planck length, and also propose a renormalization prescription to make the Wheeler-DeWitt equation finite. Some of our results have been reported in Ref.[13]. We shall give a full detail of Ref.[13] and results to the next order approximation. We shall also discuss various problems arising in our approach.

This paper is organized as follows: In Sec. II, we explain our renormalization prescription for the Wheeler-DeWitt equation. In Sec. III, we check consistency of the constraints with our renormalization prescription. In Sec. IV, we explain that the strong coupling expansion is well-suited to study the physics beyond the Planck scale. In Sec. V, we look for solutions of the Wheeler-DeWitt equation to the first order in the strong coupling expansion. In Sec. VI, we solve the Wheeler-DeWitt equation to the second order in the strong coupling expansion and discuss a problem arising in higher order calculations. Sec. VII is devoted to conclusion. In three Appendices A, B and C, a detail of computations of results in Sec. II and III are given.

II REGULARIZATION PRESCRIPTION

The (unregulated) Wheeler-DeWitt equation without matter is

$$\left[G_{ijkl}(x) \frac{\delta}{\delta h_{kl}(x)} \frac{\delta}{\delta h_{ij}(x)} - \frac{1}{(16\pi G)^2} \sqrt{h(x)} \Big(R(x) + 2\Lambda \Big) \right] \Psi[h] = 0 \ , \qquad (2.1)$$

where G is the Newton constant and G_{ijkl} is the metric on superspace

$$G_{ijkl} = \frac{1}{2\sqrt{h}} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}) \ . \qquad (2.2)$$

The $R(x)$ denotes the scalar curvature constructed from the three-metric h_{ij} and Λ is the cosmological constant. The Wheeler-DeWitt equation needs regularization because it contains a product of two functional derivatives at the same spatial point,

$$\Delta(x) \equiv G_{ijkl}(x) \frac{\delta}{\delta h_{kl}(x)} \frac{\delta}{\delta h_{ij}(x)} \ . \qquad (2.3)$$

For example, $\Delta(x)$ acting on $R(y)$ is proportional to $(\delta(x,y))^2$, which is meaningless. To make (2.1) well defined, we want to replace $\Delta(x)$ by a renormalized operator $\Delta_R(x)$. We will require $\Delta_R(x)$ to be a finite operator preserving three-dimensional

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general coordinate invariance and also to be consistent with the constraints which are the generators of the symmetry of the theory.

Recently Mansfield proposed a renormalization scheme to solve the Schrödinger equation for Yang-Mills theory in the strong coupling expansion[14]. We shall generalize the renormalization procedure developed by Mansfield to the Wheeler-DeWitt equation. Our renormalization prescription consists of two steps: The first step is to regularize the operator (2.3) by point-splitting the functional derivatives. The second step is to remove a cutoff by using analytic continuation and to extract finite quantities.

A. Regularization scheme

The first step to construct $\Delta_R(x)$ is to "point-split" the functional derivatives by use of a heat kernel[14]. We replace $\Delta(x)$ by the following differential operator:

$$\Delta(x;t) \equiv \int d^3x' K_{i'j'kl}(x',x;t)\frac{\delta}{\delta h_{kl}(x)}\frac{\delta}{\delta h_{i'j'}(x')} \ , \tag{2.4}$$

where $K_{i'j'kl}(x',x;t)$ is a bi-tensor at both x' and x and satisfies the heat equation,

$$-\frac{\partial}{\partial t}K_{i'j'kl}(x',x;t) = -(\nabla'_p\nabla'^p + \xi R(x'))K_{i'j'kl}(x',x;t) \ , \tag{2.5}$$

with the initial condition

$$\lim_{t\to 0} K_{i'j'kl}(x',x;t) = G_{i'j'kl}(x)\delta(x',x) \ . \tag{2.6}$$

Here, ∇'_p and $\delta(x',x)$ denote the covariant derivative with respect to x' and the three-dimensional δ function, respectively, and ξ is an arbitrary constant. Taking t small but nonzero in (2.4) gives a regulated operator of $\Delta(x)$. (In the naive limit $t \to 0$, $\Delta(x;t)$ is reduced to $\Delta(x)$.) In Ref.[13], the heat kernel $K_{i'j'kl}$ has been assumed to satisfy (2.5) with $\xi = 0$. We will see in Sec. VI that the term ξR in (2.5) plays an important role in finding solutions of the Wheeler-DeWitt equation to the second order in the strong coupling expansion.

We should make a few comments on the regulated operator (2.4). We have chosen the factor ordering written in (2.4). Other choices of factor ordering will lead to different values of numerical constants of our results but will not change the qualitative features. Although the heat equation (2.5) with the initial condition (2.6) has been chosen to be consistent with the requirement of preserving three-dimensional general

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coordinate invariance, it is not a unique choice to satisfy the requirement. For example, one might add m^2 or $\eta R^2(x')$ to $\nabla'_p \nabla'^p + \xi R(x')$ in (2.5). Thus, the wave functional $\Psi[h]$ depends on the choice of the heat equation and more generally on the choice of renormalization prescriptions. A renormalization prescription can make the form of a solution very simple but other renormalization prescriptions may make it complicated[4]. This fact does not , however, necessarily mean that physical quantities would depend on the choice of the heat equation, because the wave functional itself is not a physical observable. We thus hope that physical quantities are independent of any choice of the heat equation. We will not justify it in this paper.

The heat equation (2.5) can be solved by the standard technique[16]. For our purpose we need to know only $K_{i'j'kl}(x', x; t)$ and some of its covariant derivatives in the limit $x' \to x$. We will give their explicit forms in Appendix A. Let $\mathcal{O}$ be three-dimensional integrals of local functions of h_{ij}. The action of $\Delta(x; t)$ on $\mathcal{O}$ will give an expansion in powers of t. These powers of t may be determined from general coordinate invariance and dimensional analysis (t and $\Delta(x; t)$ have mass dimension -2 and 6, respectively). For example, we have

$$\Delta(x; t) \int d^3 y \sqrt{h} = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\alpha_1}{t^{3/2}} + O(t^{-1/2}) \right\}, \tag{2.7}$$

$$\Delta(x; t) \int d^3 y \sqrt{h}\, R = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\beta_1}{t^{5/2}} + \frac{\beta_2}{t^{3/2}} R(x) + O(t^{-1/2}) \right\}, \tag{2.8}$$

$$\Delta(x; t) \int d^3 y \sqrt{h}\, R^2 = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\gamma_1}{t^{7/2}} + \frac{\gamma_2}{t^{5/2}} R(x) \right.$$
$$\left. + \frac{1}{t^{3/2}} \Big(\gamma_3\, R^2(x) + \gamma_4\, R_{ij}(x)\, R^{ij}(x) + \gamma_5 \nabla_i \nabla^i R(x) \Big) + O(t^{-1/2}) \right\}, \tag{2.9}$$

$$\Delta(x; t) \int d^3 y \sqrt{h}\, R_{ij} R^{ij} = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\gamma'_1}{t^{7/2}} + \frac{\gamma'_2}{t^{5/2}}\, R(x) \right.$$
$$\left. + \frac{1}{t^{3/2}} \Big(\gamma'_3 R^2(x) + \gamma'_4\, R_{ij}(x)\, R^{ij}(x) + \gamma'_5 \nabla_i \nabla^i\, R(x) \Big) + O(t^{-1/2}) \right\}. \tag{2.10}$$

The first few coefficients are given by

$$\alpha_1 = -\frac{21}{8} \quad , \quad \beta_1 = \frac{3}{2} \quad , \quad \beta_2 = -\frac{11}{24} + \frac{3}{2} \xi \quad , \tag{2.11}$$

$$\gamma_1 = 0 \ , \quad \gamma_2 = -3 \ , \quad \gamma_3 = \frac{11}{8} - 3\,\xi \ , \quad \gamma_4 = \frac{31}{6} \ , \quad \gamma_5 = -1 \ ,$$

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[4] A similar situation occurs in gauge theories. A wave functional for non-abelian gauge theories quite depends on the choice of gauge fixings. Wilson and co-workers have recently proposed that the light-cone gauge is well-suited to study nonperturbative dynamics of QCD at low energies[15]

$$\gamma'_1 = \frac{15}{4} \; , \quad \gamma'_2 = -\frac{13}{8} + \frac{15}{4} \, \xi \; , \quad \gamma'_3 = -\frac{241}{480} - \frac{13}{8}\xi + \frac{15}{8} \, \xi^2 \; ,$$
$$\gamma'_4 = \frac{97}{20} \; , \quad \gamma'_5 = -\frac{11}{30} - \frac{25}{24} \, \xi \; . \tag{2.12}$$

The coefficients α_1, β_1 and β_2 agree with those in Ref.[13] with $\xi = 0$. A detail of the computations to get the coefficients (2.11) and (2.12) will be found in Appendix B. The above results will be used later in the discussion of finding solutions to the Wheeler-DeWitt equation in the strong coupling expansion.

B. Analytic continuation

The second step of our renormalization prescription is to extract a finite part from $\Delta(x; t = 0)\mathcal{O}$. Note that we cannot simply take the limit $t \to 0$ in $\Delta(x;t)\mathcal{O}$ because of the presence of inverse powers of t (see (2.7) to (2.10)). We will here define $\Delta_R(x)\mathcal{O}$ from $\Delta(x;t)\mathcal{O}$ by analytic continuation so that $\Delta_R(x)\mathcal{O}$ is identical to $\Delta(x; t = 0)\mathcal{O}$ (and hence $\Delta(x)\mathcal{O}$) if $\Delta(x; t = 0)\mathcal{O}$ is finite. Our definition of $\Delta_R(x)\mathcal{O}$ is given by

$$\Delta_R(x)\mathcal{O} \; \equiv \; \lim_{s \to +0} \; s \int_0^\infty d\varepsilon \, \varepsilon^{s-1} \phi(\varepsilon) \Delta(x; t = \varepsilon^2) \, \mathcal{O} \quad . \tag{2.13}$$

It is easy to see that $\Delta_R(x)\mathcal{O}$ is equal to $\Delta(x; t = 0)\mathcal{O}$ if $\Delta(x; t = 0)\mathcal{O}$ is finite as long as a differentiable function $\phi(\varepsilon)$ rapidly decreases to zero at infinity with $\phi(0) = 1$,

$$\begin{cases} \phi(\infty) & = \quad 0 \quad , \\ \phi(0) & = \quad 1 \quad . \end{cases} \tag{2.14}$$

By analytic continuation, we can give a meaning to the right hand side of (2.13) even if $\Delta(x; \varepsilon^2)\mathcal{O}$ diverges at the origin like ε^{-n} with integer n: The integral (2.13) exists for $s > n$ (provided $\phi(\varepsilon)\Delta(x; \varepsilon^2)\mathcal{O}$ has no other divergences) so that we can analytically continue s from $s > n$ to small values and take the limit $s \to 0$ to obtain a finite result[5]. For example, our definition of $\Delta_R(x)\mathcal{O}$ for $\Delta(x; \varepsilon^2)\mathcal{O} = f(x)\varepsilon^{-n}$ gives

$$\Delta_R(x)\mathcal{O} = \begin{cases} 0 & \text{for} \quad n = -1, -2, -3, \cdots \; , \\ \dfrac{1}{n!} \dfrac{d^n\phi(0)}{d\varepsilon^n} f(x) & \text{for} \quad n = 0, 1, 2, \cdots \; . \end{cases} \tag{2.15}$$

In this stage, the function $\phi(\varepsilon)$ is arbitrary as long as it satisfies the conditions (2.14). As we will see in the next section, $\phi(\varepsilon)$ has to be subject to a further condition $\frac{d\phi(0)}{d\varepsilon} = 0$. For definiteness, we may choose $\phi(\varepsilon)$ to be of the form

$$\phi(\varepsilon) \; = \; (1 + \mu\varepsilon) \, \mathrm{e}^{-\mu\varepsilon} \tag{2.16}$$

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[5] For the detail, see Ref [14].

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which satisfies all requirements, though it is not necessary to choose $\phi(\varepsilon)$ as above. Here, μ is an arbitrary mass parameter. Then, the equations (2.7) to (2.10) are now replaced by

$$\Delta_R(x) \int d^3y \sqrt{h} = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\phi^{(3)}(0)}{3!} \alpha_1 \right\} , \qquad (2.17)$$

$$\Delta_R(x) \int d^3y \sqrt{h} \, R = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\phi^{(5)}(0)}{5!} \beta_1 + \frac{\phi^{(3)}(0)}{3!} \beta_2 \, R(x) \right\} , \qquad (2.18)$$

$$\Delta_R(x) \int d^3y \sqrt{h} \, R^2 = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\phi^{(7)}(0)}{7!} \gamma_1 + \frac{\phi^{(5)}(0)}{5!} \gamma_2 \, R(x) \right.$$
$$\left. + \frac{\phi^{(3)}(0)}{3!} \Big(\gamma_3 \, R^2(x) + \gamma_4 \, R_{ij}(x) \, R^{ij}(x) + \gamma_5 \nabla_i \nabla^i \, R(x) \Big) \right\} , \qquad (2.19)$$

$$\Delta_R(x) \int d^3y \sqrt{h} \, R_{ij} \, R^{ij} = \frac{\sqrt{h(x)}}{(4\pi)^{3/2}} \left\{ \frac{\phi^{(7)}(0)}{7!} \gamma'_1 + \frac{\phi^{(5)}(0)}{5!} \gamma'_2 R(x) \right.$$
$$\left. + \frac{\phi^{(3)}(0)}{3!} \Big(\gamma'_3 \, R^2(x) + \gamma'_4 \, R_{ij}(x) \, R^{ij}(x) + \gamma'_5 \nabla_i \nabla^i \, R(x) \Big) \right\} , \qquad (2.20)$$

where

$$\phi^{(n)}(0) \equiv \frac{d^n \phi(0)}{d\varepsilon^n} = (-1)^{n-1}(n-1)\mu^n \quad . \qquad (2.21)$$

The results (2.17) to (2.20) depend on the arbitrary mass parameter μ. This is an inevitable consequence of isolating finite quantities from divergent ones. (For instance, in the dimensional regularization[17] an arbitrary mass parameter is introduced for coupling constants to have proper dimensions.) Physical observables must be independent of this arbitrary parameter μ. This leads to a renormalization group equation so that coupling "constants" should be regarded as functions of μ. This is the basic problem of renormalization. We shall return to this point in Sec. V and explicitly show that the μ-dependence can be absorbed into the redefinition of the Newton constant G and the cosmological constant Λ to the first order in the strong coupling expansion.

III CONSISTENCY OF CONSTRAINTS

We have chosen the renormalization prescription to preserve three-dimensional general coordinate invariance but this is not enough to preserve the whole symmetry of the theory at the quantum level. We have to check that our renormalization prescription would be consistent with the constraints which are the generators of the symme-

try. Consistency of the constraints requires that commutators of the constraints lead to no new constraints.

The constraints consist of the momentum constraint $\mathcal{H}_i(x)$ and the Hamiltonian constraint $\mathcal{H}(x)$:

$$\mathcal{H}_i(x) \equiv -2h_{ij}\nabla_k\pi^{jk}(x) , \tag{3.1}$$

$$\mathcal{H}(x) \equiv 16\pi G\, G_{ijkl}(x)\pi^{kl}(x)\pi^{ij}(x) + \frac{1}{16\pi G}\sqrt{h(x)}\big(R(x) + 2\Lambda\big) , \tag{3.2}$$

where $\pi^{ij}(x) = -i\frac{\delta}{\delta h_{ij}(x)}$ is the momentum operator. We will take the factor ordering such that π's stand to the right and h's stand to the left, as written above. The momentum constraints $\mathcal{H}_i$'s are the generators of three-dimensional general coordinate transformations. Since our renormalization prescription preserves three-dimensional general coordinate invariance, no anomalous terms may appear in commutators with the momentum constraints.

There remains to be considered only the commutator of the Hamiltonian constraints. In our renormalization prescription, the Hamiltonian constraint (3.2) should be replaced by

$$\mathcal{H}_R(x) \equiv -16\pi G\,\Delta_R(x) + \frac{1}{16\pi G}\sqrt{h(x)}\big(R(x) + 2\Lambda\big) . \tag{3.3}$$

The commutator $[\mathcal{H}_R(x), \mathcal{H}_R(y)]$, more correctly, $[\int d^3x\,\eta_1(x)\mathcal{H}_R(x), \int d^3y\,\eta_2(y)\mathcal{H}_R(y)]$ will have the form

$$\left[\int d^3x\,\eta_1(x)\,\mathcal{H}_R(x) \;,\; \int d^3y\,\eta_2(y)\,\mathcal{H}_R(y) \right]$$
$$= i\int d^3x\, \big(\eta_1(x)(\nabla_i\,\eta_2(x)) - (\nabla_i\,\eta_1(x))\eta_2(x)\big)\mathcal{H}^i(x) \;+\; \Delta\Gamma , \tag{3.4}$$

for arbitrary scalar functions η_1 and η_2. An anomalous term $\Delta\Gamma$ could appear from the commutators of Δ_R and $\sqrt{h}\,R$. It follows from dimensional analysis and the antisymmetry under the exchange of η_1 and η_2 that $\Delta\Gamma$ is expected to be of the form[6][13].

$$\Delta\Gamma \equiv a\,\phi^{(1)}(0)\int d^3x\sqrt{h(x)}\,\big(\eta_1(x)(\nabla_i\,\eta_2(x)) - (\nabla_i\,\eta_1(x))\eta_2(x)\big)\nabla^i R(x) . \tag{3.5}$$

In Appendix C, we will compute the commutator (3.4) and find $a = -(\frac{1}{24}+\xi)/(4\pi)^{3/2}$. We therefore take

$$\phi^{(1)}(0) = 0 \tag{3.6}$$

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[6]The $\phi^{(n)}(0)$ has mass dimension n.

for consistency of the constraints with our renormalization prescription, as claimed in the previous section.

We should make a comment on the anomaly free condition $\Delta\Gamma = 0$. One may take $\xi = -\frac{1}{24}$, instead of the condition (3.6), to have $\Delta\Gamma = 0$. As mentioned in the previous section, the heat equation (2.5) is not a unique choice to satisfy our requirements and there is a great deal of freedom to modify the heat equation. We can, however, show that $\Delta\Gamma$ is proportional to $\phi^{(1)}(0)$, irrespective of what heat equation we choose. Hence, the choice (3.6) seems more universal than the choice $\xi = -\frac{1}{24}$. We will see later that the free parameter ξ is used in finding solutions to the second order in the strong coupling expansion.

It should be noticed that the above analysis of consistency of the constraints is incomplete. Although we have found an anomalous term (3.5), which leads to the condition (3.6), it does not imply that no other anomalous terms would appear. This is because we have not exactly computed the commutators of the constraints due to lack of our knowledge of the exact expressions of $\frac{\delta K_{ijkl}(x,y;t)}{\delta h_{mn}(z)}$ and $\frac{\delta^2 K_{ijkl}(x,y;t)}{\delta h_{mn}(z)\delta h_{pq}(z')}$ for arbitrary x, y, z and z'. The exact computations of the commutators, $[\mathcal{H}_i(x), \mathcal{H}_R(y)]$ and $[\mathcal{H}_R(x), \mathcal{H}_R(y)]$, remain to be performed. If other anomalous terms were shown to appear, we have to impose further conditions besides (3.6) to be anomaly free.

IV STRONG COUPLING EXPANSION

As discussed in the previous sections, we have the renormalized Wheeler-DeWitt equation,

$$\left[\Delta_R(x) - \frac{1}{(16\pi G)^2} \sqrt{h(x)}\big(R(x) + 2\Lambda \big) \right] \Psi[h] = 0 , \tag{4.1}$$

with the condition (3.6). It is not easy to solve the full Wheeler-DeWitt equation exactly. Since we are interested in short distance behavior of the Wheeler-DeWitt equation, we do not probably need to solve (4.1) exactly. The Planck length is given by $l_p = (\hbar G/c^3)^{1/2}$ so that the strong coupling limit, i.e., $G \to \infty$, will be well-suited to probe quantum geometry at length scales much *smaller* than the Planck length[18]. We will discuss this point in some detail later. Although strong coupling quantum gravity has been studied before, their studies are not satisfactory to clarify quantum geometry at short distances because little attention has been paid to the regularization of the Wheeler-DeWitt equation.

To solve the equation (4.1) with the condition (3.6), we attempt an expansion of the wave functional of the universe in inverse powers of the Newton constant G. We will first rewrite the Wheeler-DeWitt equation (4.1) as

$$\Delta_R(x)\, S[h] - G_{ijkl}(x)\frac{\delta S[h]}{\delta h_{kl}(x)}\frac{\delta S[h]}{\delta h_{ij}(x)} + \frac{1}{(16\pi G)^2}\sqrt{h(x)}\big(R(x) + 2\Lambda\big) = 0 \;, \qquad (4.2)$$

where

$$\Psi[h] \equiv \exp\{-S[h]\} \;. \qquad (4.3)$$

We then assume that the functional $S[h]$ has the form

$$S[h] = \sum_{n=0}^{\infty}\Big(\frac{1}{16\pi G}\Big)^{2n} S_n[h] \;. \qquad (4.4)$$

Substituting (4.4) into (4.2), we have , according to the inverse powers of G,

$$\Delta_R(x)\, S_0[h] - G_{ijkl}(x)\frac{\delta S_0[h]}{\delta h_{kl}(x)}\frac{\delta S_0[h]}{\delta h_{ij}(x)} = 0 \;, \qquad (4.5)$$

$$\Delta_R(x)\, S_1[h] - 2\, G_{ijkl}(x)\frac{\delta S_0[h]}{\delta h_{kl}(x)}\frac{\delta S_1[h]}{\delta h_{ij}(x)} + \sqrt{h(x)}\big(R(x) + 2\Lambda\big) = 0 \;, \qquad (4.6)$$

$$\Delta_R(x)\, S_n[h] - G_{ijkl}(x)\sum_{m=0}^{n}\frac{\delta S_m[h]}{\delta h_{kl}(x)}\frac{\delta S_{n-m}[h]}{\delta h_{ij}(x)} = 0 \;, \qquad n = 2,3,4\cdots. \quad (4.7)$$

To solve the Wheeler-DeWitt equation in the strong coupling expansion, we adopt an ansatz of locality[14]. The functional $S_n[h]$ is assumed to be a sum of integrals of local functions of h_{ij}. There seem no obvious reasons to restrict our attention to a class of local solutions because non-locality is all over in this theory. A main reason why we adopt the locality ansatz is technical difficulties to find general solutions. Although physical significance of local solutions are quite unclear, it may worth while studying them to see how our formulation works.

Under the locality ansatz we find two solutions to the zeroth order equation (4.5), i.e.,

$$S_0[h] = 0 \;, \qquad (4.8)$$

and

$$S_0[h] = \frac{7\mu^3}{3(4\pi)^{3/2}}\int d^3x\sqrt{h(x)} \;. \qquad (4.9)$$

(To see this, we can use the relation (2.17) and (2.21).)

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We would like to make two comments on the expansion (4.4). In Ref.[13], the expansion in (4.4) has been assumed to begin with $n = 1$, which corresponds to the first solution (4.8). We here found another solution, though it turns out that the second solution (4.9) leads to essentially the same solution to higher orders as the first one. More generally, one might expand $S[h]$ as

$$\sum_{n=-m}^{\infty} \left(\frac{1}{16\pi G}\right)^{2n} S_n[h] \tag{4.10}$$

with some positive integer m. To leading order, we would have

$$G_{ijkl}(x) \frac{\delta S_{-m}[h]}{\delta h_{kl}(x)} \frac{\delta S_{-m}[h]}{\delta h_{ij}(x)} = 0 \, , \tag{4.11}$$

which will lead to $S_{-m} = 0$ under the locality ansatz. Then, S_{-m+1} is found to satisfy the same equation (4.11) with the replacement $-m \to -m + 1$. Repeating the same step above, we finally conclude that $S_{-m} = 0$ for $m \geq 1$. The second comment concerns the powers of the Newton constant G in the expansion (4.4). We have assumed that the wave functional is expanded in powers of G^{-2}. We can, however, assume that the wave functional is more generally expanded in powers of G^{-1}, which will still give a consistent expansion of the equation (4.2). In this paper, we will restrict our considerations to the expansion (4.4) since (4.4) respects a symmetry $(G \to -G)$ of the Wheeler-DeWitt equation, but the generalization to a power series of G^{-1} is straightforward.

Before closing this section, we would like to discuss physical meanings of the strong coupling expansion in more detail. The semiclassical expansion corresponds to the expansion in powers of $\hbar$, while the strong coupling expansion (4.4) corresponds to the expansion in powers of $\hbar^{-2}$ because $\hbar$ appears in the combination of $\hbar G$ in quantum gravity. Thus the wave functional in the strong coupling expansion is expected to describe quite different physics from the semiclassical one. The semiclassical approximation will be valid for long-wavelength gravitational fields, while the strong coupling approximation may be useful when the wavelength of gravitational fields very rapidly changes in a wavelength.

Although we have mentioned that the strong coupling expansion is well-suited to study the physics at length scales much smaller than the Planck length, we want to discuss this point more precisely in our framework. We will here consider the first solution (4.8) to the zeroth order, i.e., $S_0[h] = 0$. For the strong coupling expansion

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(4.4) to be meaningful, the successive terms in the series for $S[h]$ should satisfy, in particular,

$$\left(\frac{1}{16\pi G}\right)^2 S_1[h] \gg \left(\frac{1}{16\pi G}\right)^4 S_2[h] \quad . \tag{4.12}$$

Taking $\Lambda = 0$ for simplicity and using $\phi^{(n)}(0) \sim \mu^n$, we can write the first order term as

$$\left(\frac{1}{16\pi G}\right)^2 S_1[h] = \left(\frac{1}{G\mu^2}\right)^2 \int d^3x \sqrt{h(x)} \mu^3 \left\{ c_1 + c_2 \frac{R(x)}{\mu^2} \right\} \quad , \tag{4.13}$$

as we will find in the next section. The c_n's are dimensionless constants of order one. The second order term will be shown, in Sec. VI, to take the form

$$\left(\frac{1}{16\pi G}\right)^4 S_2[h] = \left(\frac{1}{G\mu^2}\right)^4 \int d^3x \sqrt{h(x)} \mu^3$$
$$\times \left\{ c'_1 + c'_2 \frac{R(x)}{\mu^2} + \frac{1}{\mu^4}\left(c'_3 R^2(x) + c'_4 R_{ij}(x) R^{ij}(x) \right) \right\} , \tag{4.14}$$

where c'_n's are dimensionless constants of order one. It follows that we can drop the second order term $S_2[h]$, provided that $G\mu^2 \gg 1$ (i.e., $\mu \gg m_p$), $R \sim \mu^2$ and $R_{ij}R^{ij} \sim \mu^4$. We therefore expect that the wave functional to the first order term well describes the universe with a curvature much larger than the Planck mass or with a radius much smaller than the Planck length. It should be emphasized that the strong coupling expansion is not a derivative expansion because higher derivative terms like $(\frac{R}{\mu^2})^m (m \geq 2)$ could appear on the right hand side of (4.13) but it happens that their coefficients are zero as a solution to the equation (4.6).

If we apply the strong coupling expansion (4.4) to a minisuperspace model, we can see that the expansion is essentially identical to a small radius expansion of the universe. Thus we may expect that our expansion gives a good approximation for a universe with a radius much smaller than the Planck length, though the minisuperspace model will not be applicable to this region.

V   THE FIRST ORDER SOLUTIONS

In this section, we shall look for solutions to $S_1[h]$ in the strong coupling expansion. Since both zeroth order solutions (4.8) and (4.9) lead to essentially the same solution to $S_1[h]$, we will mainly consider the zeroth order solution (4.8). Substituting $S_0[h] = 0$ into the first order equation (4.6), we get

$$\Delta_R(x) S_1[h] = -\sqrt{h(x)} \left(R(x) + 2\Lambda \right) . \tag{5.1}$$

From (2.17) and (2.18), we find a solution to the first order as[7]

$$S_1[h] = \int d^3x \sqrt{h(x)} \left\{ a_1 + a_2 R(x) \right\} , \tag{5.2}$$

where

$$a_1 = \frac{16(4\pi)^{3/2}}{7\mu^3} \left\{ \Lambda + \frac{9\mu^2}{5(11 - 36 \; \xi)} \right\} ,$$

$$a_2 = \frac{72(4\pi)^{3/2}}{(11 - 36 \; \xi)\mu^3} . \tag{5.3}$$

This result agrees with the one obtained in Ref.[13] with $\xi = 0$. One may try to find other solutions to $S_1[h]$. It turns out that other solutions to $S_1[h]$, if exist, may be given by the integral of nonlocal functions of h_{ij}, i.e., $S_1[h]$ includes infinitely many higher derivative terms.

We would like to briefly discuss the renormalizability. In our formulation, the renormalizability of the theory requires that all physical quantities must be independent of the arbitrary mass parameter μ, so that the Newton "constant" G and the cosmological "constant" Λ should be regarded as functions of μ. To the first order, the above statement may be replaced by saying that the wave functional is independent of μ. This is achieved by requiring that the following combinations are independent of μ:

$$\left. \begin{array}{c} G^2(\mu) \; \mu^3 \\[2mm] \Lambda(\mu) + \dfrac{9 \; \mu^2}{5(11 - 36 \; \xi)} \end{array} \right\} \mu - \text{independent} \quad . \tag{5.4}$$

Thus the μ-dependence can be absorbed into the redefinition of G and Λ to the first order in the strong coupling expansion. The above observation (5.4) is a good news for our strong coupling expansion, which is expected to give a good approximation scheme for large μ as discussed in the previous section, because the actual dimensionless expansion parameter $(G^2(\mu)\mu^4)^{-1}$ tends to zero as the mass scale μ increases.

We will finally look for a solution to $S_1[h]$ in the case of the second solution (4.9). Then, the first order equation (4.6) is reduced to

$$\Delta_R(x) \, S_1[h] + \frac{7\mu^3}{6(4\pi)^{3/2}} \frac{\delta S_1[h]}{\delta h_i{}^i(x)} = -\sqrt{h(x)}\Big(R(x) + 2\Lambda \Big) . \tag{5.5}$$

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[7] In Ref.[19], Kodama has pointed out that the exponential of the Chern-Simons action, which is equivalent to the Einstein action[20], is an exact solution of the Hamiltonian constraint in the holomorphic representation of the Ashtekar formalism[8]. Connections with our solution are unclear.

It is easy to show that a solution to (5.5) is given by

$$S_1[h] = \int d^3x \sqrt{h(x)}\, \{a'_1 + a'_2 R(x)\} \qquad (5.6)$$

where

$$a'_1 = -\frac{16(4\pi)^{3/2}}{7\mu^3}\left[\Lambda - \frac{9\mu^2}{5(31 + 36\ \xi)}\right] ,$$

$$a'_2 = -\frac{72(4\pi)^{3/2}}{(31 + 36\ \xi)\mu^3} . \qquad (5.7)$$

The difference between the solutions (5.2) and (5.6) lies only in the numerical coefficients and hence all the discussions on the solution (5.2) will remain true for the solution (5.6).

VI THE SECOND ORDER SOLUTIONS

In the previous section, we have found the first order solution for $S_1[h]$ in the strong coupling expansion. For our expansion scheme to be consistent, we need to show that $S_n[h]$ for $n \geq 2$ can in principle be constructed order by order and also that the successive terms obey, in particular, the relation (4.12); otherwise our solutions are meaningless. To this end, in this section we look for solutions to the second order equation

$$\Delta_R(x)\, S_2[h] - G_{ijkl}(x)\left\{2\frac{\delta S_0[h]}{\delta h_{kl}(x)}\frac{\delta S_2[h]}{\delta h_{ij}(x)} + \frac{\delta S_1[h]}{\delta h_{kl}(x)}\frac{\delta S_1[h]}{\delta h_{ij}(x)}\right\} = 0 , \qquad (6.1)$$

and show that $S_2[h]$ has the form (4.14), as announced in Sec.IV. We also discuss a problem of finding higher order solutions. The equation (6.1) may be solved by assuming $S_2[h]$ to take the form

$$S_2[h] = \int d^3x \sqrt{h(x)}\{b_1 + b_2 R(x) + b_3 R^2(x) + b_4 R_{ij}(x)R^{ij}(x)\} . \qquad (6.2)$$

Note that the Riemann curvature R_{ijkl} can be expressed in terms of R and R_{ij} in three-dimensions and also that the term $\int d^3x \sqrt{h}\nabla_i \nabla^i R$ is not included in $S_2[h]$ because it vanishes identically (without boundaries). Substitution of (4.8), (5.2) and (6.2) into (6.1) leads to

$$B_1 + B_2 R(x) + B_3 R^2(x) + B_4 R_{ij}(x)R^{ij}(x) + B_5 \nabla_i \nabla^i R(x) = 0 , \qquad (6.3)$$

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where

$$B_1 = b_1\alpha_1\frac{\mu^3}{3(4\pi)^{3/2}} + b_2\beta_1\frac{\mu^5}{30(4\pi)^{3/2}} + (b_3\gamma_1 + b_4\gamma'_1)\frac{\mu^7}{840(4\pi)^{3/2}} + \frac{3}{8}(a_1)^2 \ ,$$

$$B_2 = b_2\beta_2\frac{\mu^3}{3(4\pi)^{3/2}} + (b_3\gamma_2 + b_4\gamma'_2)\frac{\mu^5}{30(4\pi)^{3/2}} + \frac{1}{4}a_1a_2 \ ,$$

$$B_3 = (b_3\gamma_3 + b_4\gamma'_3)\frac{\mu^3}{3(4\pi)^{3/2}} + \frac{3}{8}(a_2)^2 \ , \tag{6.4}$$

$$B_4 = (b_3\gamma_4 + b_4\gamma'_4)\frac{\mu^3}{3(4\pi)^{3/2}} - (a_2)^2 \ ,$$

$$B_5 = (b_3\gamma_5 + b_4\gamma'_5)\frac{\mu^3}{3(4\pi)^{3/2}} \ .$$

The equation (6.3) or equivalently the five equations

$$B_n = 0 \ , \qquad \text{for} \quad n = 1, 2, \cdots, 5 \ , \tag{6.5}$$

will, in general, have no solutions because the number of the free parameters ($b_m, m = 1, \cdots, 4$) is less than the number of the equations ($B_n, n = 1, \cdots, 5$). This is due to the fact that the term $\nabla_i\nabla^i R(x)$ appears in (6.3) but not in (6.2), in other words, the Wheeler-DeWitt equation is a local equation imposed by the Hamiltonian density (not the Hamiltonian itself), while $S_2[h]$ is assumed to be the three-dimensional integral of local functions[8]. Hence we need one more parameter to solve the equation (6.3) or (6.5). To this end, we will here use the arbitrariness of defining the kernel $K_{i'j'kl}$. It is not difficult to show that the equation (6.3) or (6.5) has a solution, provided that ξ is chosen to be one of the solutions to the equation

$$9600\ \xi^2 - 7633\ \xi + 196 = 0 \ . \tag{6.6}$$

To solve higher order equations for $S_n[h](n \geq 3)$ with the ansatz of locality, we will face a similar problem and need to generalize the heat equation (2.5) to include more arbitrary parameters. Thus, in the strong coupling expansion the form of the wave functional to the second or higher order crucially depends on the choice of the heat equation. We do not know whether or not this fact causes serious problems in our formulation because the wave functional itself is not a physical observable and because it is inevitable that the wave functional depends on the renormalization

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[8] This kind of problems does not occur for Yang-Mills theory because the Hamiltonian appears in the Schrödinger equation.

prescription. Our hope is that physical quantities can be independent of our renormalization prescription and that our solutions have important physical meanings with a special choice of the heat equation.

In the case of the second solution (4.9) to the zeroth order, we can also show that the second order equation (6.1) with the ansatz (6.2) has a solution, provided that ξ is chosen to be one of the solutions to the equation

$$40800\ \xi^2 - 9307\ \xi + 1834 = 0\ . \tag{6.7}$$

Since the discriminant is negative, the solutions are complex. This implies that an imaginary part appears in the renormalized Hamiltonian operator through the kernel $K_{i'j'kl}$, so the second solution (4.9) to the zeroth order seems to lead to an undesirable result. Since the hermiticity of the Hamiltonian is unclear in quantum gravity due to the lack of the knowledge of the functional measure $\mathcal{D}h$, we will not discuss the problem furthermore.

VII   CONCLUSION

We have proposed a renormalization prescription to the Wheeler-DeWitt equation and solved it to the second order in the strong coupling expansion to study quantum geometry at length scales much smaller than the Planck length. We have restricted our attention to a class of local solutions for mainly technical reasons. Our formulation is not, however, limited to a class of local solutions and our renormalization prescription does not rely on the strong coupling expansion. It would be challenging to look for non-local solutions in the strong coupling expansion and for exact (local or non-local) solutions without the expansion[21].

We have also restricted our attention to pure gravity. Recently, the dilaton gravity

$$\mathcal{L} = \sqrt{g}\ \mathrm{e}^{-2\Phi}\Big(R - 4D_\mu\Phi D^\mu\Phi + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)$$

has extensively been studied to reveal stringy phenomena[22]. It would be of great interest to investigate the theory in our formulation to reveal stringy dynamics beyond the Planck scale.

APPENDIX A: THE HEAT KERNEL IN THE COINCIDENCE LIMIT

In this appendix, we will compute the heat kernel $K_{i'j'kl}$ and its covariant derivatives $K_{i'j'kl;a'}$, $K_{i'j'kl;a'b'}$ and $K_{i'j'kl;a'b'c'd'}$ in the coincidence limit $x' \to x$.

According to the standard technique[16], the heat equation (2.5) with the initial condition (2.6) can be solved by the ansatz

$$K_{i'j'kl}(x', x; t) = \frac{(\Delta(x', x))^{1/2}}{(4\pi t)^{3/2}} \, e^{-\frac{\sigma(x', x)}{2t}} \sum_{n=0}^{\infty} a_{i'j'kl}^{(n)}(x', x) \, t^n \,, \tag{A.1}$$

where the bi-scalar $\sigma(x', x)$ is the geodesic integral equal to one half the square of the distance along the geodesic between x' and x, and the bi-scalar $\Delta(x', x)$ is defined by[9]

$$\Delta(x', x) = h(x')^{-\frac{1}{2}} \, \det(\sigma_{;i'j}(x', x)) \, h(x)^{-\frac{1}{2}} \,. \tag{A.2}$$

The properties of $\sigma(x', x)$ and $\Delta(x', x)$ have been discussed in detail in Ref.[23]. We will not repeat the discussions here.

The initial condition (2.6) implies that

$$\lim_{x' \to x} a_{i'j'kl}^{(0)} = \sqrt{h} G_{i'j'kl} \,. \tag{A.3}$$

Inserting the ansatz (A.1) into the equation (2.5), one finds

$$\sigma^{;p'}(x', x) a_{i'j'kl;p'}^{(0)}(x', x) = 0 \,, \tag{A.4}$$

$$n \, a_{i'j'kl}^{(n)}(x', x) + \sigma^{;p'}(x', x) \, a_{i'j'kl;p'}^{(n)}(x', x)$$

$$= \left(\Delta(x', x) \right)^{-\frac{1}{2}} \left((\Delta(x', x))^{\frac{1}{2}} a_{i'j'kl}^{(n-1)}(x', x) \right)^{;p'}_{p'} + \xi R(x') a_{i'j'kl}^{(n-1)}(x', x) \,, \tag{A.5}$$

$$\text{for} \quad n = 1, 2, \cdots .$$

For our purpose, we need to know values for the first few a's in the expansion (A.1) and some of their covariant derivatives in the limit $x' \to x$. They can be obtained by repeatedly differentiating the equations (A.4) and (A.5) and then by taking the limit $x' \to x$. The results are

$$\lim_{x' \to x} a_{i'j'kl}^{(0)} \qquad = \quad \sqrt{h} G_{i'j'kl} \quad,$$

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[9] The $\sigma_{;i'j}(x', x)$ denotes the covariant derivatives of $\sigma(x', x)$ with respect to x'^i and x^j. The bi-scalar $\Delta(x', x)$ should not be confused with the differential operator $\Delta(x)$ or $\Delta(x; t)$ in the text.

$$\lim_{x'\to x} a^{(0)}_{i'j'kl;a'} \quad = \quad 0 \quad,$$

$$\lim_{x'\to x} a^{(0)}_{i'j'kl;a'b'} \quad = \quad \sqrt{h}\, G_{p'j'kl}\Big(-\frac{1}{2} R^{p'}{}_{i'a'b'}\Big) + (i' \leftrightarrow j') \quad,$$

$$\lim_{x'\to x} a^{(0)}_{i'j'kl;a'b'c'} \quad = \quad \sqrt{h}\, G_{p'j'kl}\Big(-\frac{1}{3} R^{p'}{}_{i'a'b';c'} - \frac{1}{3} R^{p'}{}_{i'a'c';b'}\Big) + (i' \leftrightarrow j') \quad,$$

$$\lim_{x'\to x} a^{(0)}_{i'j'kl;a'b'c'd'} \quad = \quad \sqrt{h}\, G_{p'q'kl}\Big(\frac{1}{4}(R^{p'}{}_{i'a'b'} R^{q'}{}_{j'c'd'} + R^{p'}{}_{i'a'c'} R^{q'}{}_{j'b'd'} + R^{p'}{}_{i'a'd'} R^{q'}{}_{j'b'c'}$$
$$+ R^{p'}{}_{i'c'd'} R^{q'}{}_{j'a'b'} + R^{p'}{}_{i'b'd'} R^{q'}{}_{j'a'c'} + R^{p'}{}_{i'b'c'} R^{q'}{}_{j'a'd'})\Big)$$

$$+ \Big\{ \sqrt{h}\, G_{pj'kl}\Big(\frac{1}{12}(R^{p'}{}_{i'q'a'} R^{q'}{}_{b'd'c'} + R^{p'}{}_{i'q'a'} R^{q'}{}_{d'b'c'}$$
$$+ R^{p'}{}_{i'q'b'} R^{q'}{}_{a'c'd'} + R^{p'}{}_{i'q'b'} R^{q'}{}_{c'a'd'}$$
$$+ R^{p'}{}_{i'q'c'} R^{q'}{}_{a'b'd'} + R^{p'}{}_{i'q'c'} R^{q'}{}_{b'a'd'}$$
$$+ R^{p'}{}_{i'q'd'} R^{q'}{}_{a'b'c'} + R^{p'}{}_{i'q'd'} R^{q'}{}_{b'a'c'})$$

$$+ \frac{1}{8}(R^{q'}{}_{i'a'b'} R^{p'}{}_{q'c'd'} + R^{q'}{}_{i'a'c'} R^{p'}{}_{q'b'd'} + R^{q'}{}_{i'a'd'} R^{p'}{}_{q'b'c'}$$
$$+ R^{q'}{}_{i'c'd'} R^{p'}{}_{q'a'b'} + R^{q'}{}_{i'b'd'} R^{p'}{}_{q'a'c'} + R^{q'}{}_{i'b'c'} R^{p'}{}_{q'a'd'})$$

$$- \frac{1}{4}\left(R^{p}{}_{i'a'b';c'd'} + R^{p}{}_{i'a'c';b'd'} + R^{p}{}_{i'a'd';b'c'}\right) + (i' \leftrightarrow j')\Big\} \quad,$$

$$\lim_{x'\to x} a^{(1)}_{i'j'kl} \quad = \quad \sqrt{h}\, G_{i'j'kl}\Big(-\frac{1}{6} + \xi\Big) R \quad,$$

$$\lim_{x'\to x} a^{(1)}_{i'j'kl;a'} \quad = \quad \sqrt{h}\, G_{i'j'kl}\Big(-\frac{1}{12} + \frac{1}{2}\,\xi\Big) R_{;a'} + \Big\{ \sqrt{h}\, G_{q'j'kl}\Big(\frac{1}{6} R^{q'}{}_{i'a'p'}{}^{;p'}\Big) + (i' \leftrightarrow j')\Big\} \quad,$$

$$\lim_{x'\to x} a^{(1)}_{i'j'kl;a'b'} \quad = \quad \sqrt{h}\, G_{i'j'kl}\Big(\frac{1}{90} R^{p'q'} R_{p'a'q'b'} - \frac{1}{45} R^{p'}{}_{a'} R_{p'b'} + \frac{1}{90} R^{p'q'm'}{}_{a'} R_{p'q'm'b'}$$
$$- \Big(\frac{1}{20} - \frac{1}{3}\,\xi\Big) R_{;a'b'} - \frac{1}{60} R_{a'b';p'}{}^{p'}\Big)$$

$$+ \sqrt{h}\, G_{p'q'kl}\Big(\frac{1}{6} R^{p'}{}_{i'm'a'} R^{q'}{}_{j'}{}^{m'}{}_{b'} + \frac{1}{6} R^{p'}{}_{j'm'a'} R^{q'}{}_{i'}{}^{m'}{}_{b'}\Big)$$

$$+ \Big\{ \sqrt{h}\, G_{p'j'kl}\Big(\Big(\frac{1}{12} - \frac{1}{2}\,\xi\Big) R R^{p'}{}_{i'a'b'} + \frac{1}{12} R_{q'i'm'a'} R^{p'q'm'}{}_{b'}$$
$$+ \frac{1}{12} R_{q'i'm'b'} R^{p'q'm'}{}_{a'} - \frac{1}{12} R^{p'}{}_{i'}{}^{q'}{}_{a';q'b'} - \frac{1}{12} R^{p'}{}_{i'}{}^{q'}{}_{b';q'a'}\Big)$$
$$+ (i' \leftrightarrow j')\Big\} \quad,$$

$$\lim_{x'\to x} a^{(2)}_{i'j'kl} \quad = \quad \sqrt{h}\, G_{i'j'kl}\Big(\Big(\frac{1}{72} - \frac{1}{6}\,\xi + \frac{1}{2}\,\xi^2\Big) R^2 - \frac{1}{180} R_{p'q'} R^{p'q'}$$
$$+ \frac{1}{180} R^{p'q'm'n'} R_{p'q'm'n'} - \Big(\frac{1}{30} - \frac{1}{6}\,\xi\Big) R_{;p'}{}^{p'}\Big)$$

$$+ \frac{1}{6}\sqrt{h}\, G_{p'q'kl} R^{p'}{}_{i'm'n'} R^{q'}{}_{j'}{}^{m'n'}$$

$$+ \Big\{\frac{1}{12}\sqrt{h}\, G_{p'j'kl} R_{q'i'm'n'} R^{p'q'm'n'} + (i' \leftrightarrow j')\Big\} \quad. \tag{A.6}$$

These results enable us to compute $K_{i'j'kl}$, $K_{i'j'kl;a'}$, $K_{i'j'kl;a'b'}$ and $K_{i'j'kl;a'b'c'd'}$ in

the limit $x' \to x$.

$$\lim_{x' \to x} K_{i'j'kl} = \frac{1}{(4\pi t)^{3/2}} \sqrt{h} G_{i'j'kl} \Big\{ 1 - t\Big(\frac{1}{6} - \xi\Big)R \Big\} + O(t^{\frac{1}{2}}) , \tag{A.7}$$

$$\lim_{x' \to x} K_{i'j'kl;a'}$$
$$= \frac{1}{(4\pi)^{3/2}t^{1/2}} \Big\{ -\frac{1}{2}\Big(\frac{1}{6} - \xi\Big)\sqrt{h} G_{i'j'kl}R_{;a'}$$
$$+ \frac{1}{6}\Big(\sqrt{h} G_{p'j'kl}R^{p'}{}_{i'a'q'}{}^{;q'} + (i' \leftrightarrow j') \Big) \Big\} + O(t^{\frac{1}{2}}) , \tag{A.8}$$

$$\lim_{x' \to x} K_{i'j'kl;a'b'}$$
$$= \frac{1}{(4\pi)^{3/2}t^{5/2}} \Big\{ -\frac{1}{2}\sqrt{h} G_{i'j'kl}h_{a'b'} \Big\}$$
$$+ \frac{1}{(4\pi t)^{3/2}} \Big\{ \sqrt{h} G_{ijkl}\Big(-\frac{1}{6}R_{a'b'} + \Big(\frac{1}{12} - \frac{1}{2}\xi\Big)h_{a'b'}R\Big)$$
$$- \frac{1}{2}\Big(\sqrt{h} G_{p'j'kl}R^{p'}{}_{i'a'b'} + (i' \leftrightarrow j') \Big) \Big\} + O(t^{-\frac{1}{2}}) , \tag{A.9}$$

$$\lim_{x' \to x} K_{i'j'kl;a'b'c'd'}$$
$$= \frac{1}{(4\pi t)^{3/2}} \Big\{ \frac{1}{t^2}F^{(0)}_{i'j'kla'b'c'd'} + \frac{1}{t}F^{(1)}_{i'j'kla'b'c'd'} + F^{(2)}_{i'j'kla'b'c'd'} \Big\} + O(t^{-\frac{1}{2}}) , \tag{A.10}$$

where

$$F^{(0)}_{i'j'kla'b'c'd'} = \frac{1}{8}\sqrt{h} G_{i'j'kl}[h_{\underline{a'b'}}h_{\underline{c'd'}}]_s ,$$

$$F^{(1)}_{i'j'kla'b'c'd'}$$
$$= \sqrt{h} G_{i'j'kl}\Big\{-\frac{1}{6}(R_{a'c'b'd'} + R_{a'd'b'c'}) + \frac{1}{12}[h_{\underline{a'b'}}R_{\underline{c'd'}}]_s - \Big(\frac{1}{48} - \frac{1}{8}\xi\Big)R \, [h_{\underline{a'b'}}h_{\underline{c'd'}}]_s \Big\}$$
$$+ \Big\{ \frac{1}{4}\sqrt{h} G_{p'j'kl}[h_{\underline{a'b'}}R^{p'}{}_{i'\underline{c'd'}}]_s + (i' \to j') \Big\} ,$$

$$F^{(2)}_{i'j'kla'b'c'd'}$$
$$= \sqrt{h} G_{i'j'kl}\Big\{ \frac{1}{180}\Big(R_{p'a'}(11R^{p'}{}_{c'b'd'} - 10R^{p'}{}_{d'b'c'}) + R_{p'b'}(11R^{p'}{}_{c'a'd'} - 10R^{p'}{}_{d'a'c'})$$
$$+ R_{p'c'}(11R^{p'}{}_{b'a'd'} - 10R^{p'}{}_{d'a'b'}) + R_{p'd'}(11R^{p'}{}_{b'a'c'} - 10R^{p'}{}_{c'a'b'})\Big)$$
$$+ \Big(\frac{1}{36} - \frac{1}{6}\xi\Big)R \, (R_{a'c'b'd'} + R_{a'd'b'c'}) + \frac{1}{72}[R_{\underline{a'b'}}R_{\underline{c'd'}}]_s$$
$$+ \frac{1}{8}\Big(\Big(\frac{1}{72} - \frac{1}{6}\xi + \frac{1}{2}\xi^2\Big)R^2 - \frac{1}{180}R^{p'q'}R_{p'q'} + \frac{1}{180}R^{p'q'm'n'}R_{p'q'm'n}$$
$$- \Big(\frac{1}{30} - \frac{1}{6}\xi\Big)R^{;p'}{}_{p'}\Big)[h_{\underline{a'b'}}h_{\underline{c'd'}}]_s - \Big(\frac{1}{72} - \frac{1}{12}\xi\Big)R \, [h_{\underline{a'b'}}R_{\underline{c'd'}}]_s$$
$$- \frac{1}{72}[R^{p'q'}{}_{\underline{a'b'}}R_{p'q'\underline{c'd'}}]_s + \frac{1}{45}[R^{p'q'}{}_{\underline{a'b'}}R_{p'\underline{c'}q'\underline{d'}}]_s + \frac{1}{90}[R^{p'}{}_{\underline{a'}}{}^{q'}{}_{\underline{b'}}R_{p'\underline{c'}q'\underline{d'}}]_s$$
$$- \frac{1}{180}[h_{\underline{a'b'}}R^{p'q'}R_{p'\underline{c'}q'\underline{d'}}]_s + \frac{1}{90}[h_{\underline{a'b'}}R^{p'}{}_{\underline{c'}}R_{p'\underline{d'}}]_s - \frac{1}{180}[h_{\underline{a'b'}}R^{p'q'm'}{}_{\underline{c'}}R_{p'q'm'\underline{d'}}]_s$$

$$-\frac{1}{20}[R_{\underline{a'b'};\underline{c'd'}}]_s + \frac{1}{120}[h_{\underline{a'b'}}R_{\underline{c'd'};p'}{}^{p'}]_s + \left(\frac{1}{40}-\frac{1}{6}\xi\right)[h_{\underline{a'b'}}R_{;\underline{c'd'}}]_s\Big\}$$

$$+\sqrt{h}G_{p'q'kl}\Big\{-\frac{1}{12}[h_{\underline{a'b'}}R^{p'}{}_{i'm'\underline{c'}}R^{q'}{}_{j'}{}^{m'}{}_{\underline{d'}}]_s - \frac{1}{12}[h_{\underline{a'b'}}R^{p'}{}_{i'm'\underline{d'}}R^{q'}{}_{j'}{}^{m'}{}_{\underline{c'}}]_s$$

$$+\frac{1}{48}R^{p'}{}_{i'}{}^{m'n'}R^{q'}{}_{j'm'n'}[h_{\underline{a'b'}}h_{\underline{c'd'}}]_s + \frac{1}{4}[R^{p'}{}_{i'\underline{a'b'}}R^{q'}{}_{j'\underline{c'd'}}]_s\Big\}$$

$$+\Big\{\sqrt{h}G_{pj'kl}\Big\{\frac{1}{12}(R^{p'}{}_{i'q'a'}R^{q'}{}_{b'd'c'} + R^{p'}{}_{i'q'a'}R^{q'}{}_{d'b'c'} + R^{p'}{}_{i'q'b'}R^{q'}{}_{a'c'd'} + R^{p'}{}_{i'q'b'}R^{q'}{}_{c'a'd'}$$

$$+R^{p'}{}_{i'q'c'}R^{q'}{}_{a'b'd'} + R^{p'}{}_{i'q'c'}R^{q'}{}_{b'a'd'} + R^{p'}{}_{i'q'd'}R^{q'}{}_{a'b'c'} + R^{p'}{}_{i'q'd'}R^{q'}{}_{b'a'c'})$$

$$-\frac{1}{4}(R^{p'}{}_{i'a'b';c'd'} + R^{p'}{}_{i'a'c';b'd'} + R^{p'}{}_{i'a'd';b'c'})$$

$$+\frac{1}{96}R^{p'q'm'n'}R_{q'i'm'n'}[h_{\underline{a'b'}}h_{\underline{c'd'}}]_s - \left(\frac{1}{24}-\frac{1}{4}\xi\right)R[h_{\underline{a'b'}}R^{p'}{}_{i'\underline{c'd'}}]_s$$

$$-\frac{1}{24}[h_{\underline{a'b'}}R^{p'q'm'}{}_{\underline{d'}}R_{q'i'm'\underline{c'}}]_s - \frac{1}{24}[h_{\underline{a'b'}}R^{p'q'm'}{}_{\underline{c'}}R_{q'i'm'\underline{d'}}]_s$$

$$+\frac{1}{12}[R_{\underline{a'b'}}R^{p'}{}_{i'\underline{c'd'}}]_s + \frac{1}{8}[R^{q'}{}_{i'\underline{a'b'}}R^{p'}{}_{q'\underline{c'd'}}]_s + \frac{1}{24}[h_{\underline{a'b'}}R^{p'}{}_{i'}{}^{q'}{}_{\underline{c'};q\underline{d'}}]_s$$

$$+\frac{1}{24}[h_{\underline{a'b'}}R^{p'}{}_{i'}{}^{q'}{}_{\underline{d'};q\underline{c'}}]_s\Big\} + (i' \leftrightarrow j')\Big\}\,, \tag{A.11}$$

$$\tag{A.12}$$

Here, we have used a notation $[X_{\underline{ab}ij\cdots}Y_{\underline{cd}kl\cdots}]_s$, which is an abbreviation of the sum of the six terms in different order with respect to the underlined indices $\underline{a}, \underline{b}, \underline{c}$ and $\underline{d}$, i.e.,

$$\begin{aligned}[X_{\underline{ab}ij}\cdots Y_{\underline{cd}kl}\cdots]_s &\equiv & X_{abij\cdots}Y_{cdkl\cdots} + X_{cdij\cdots}Y_{abkl\cdots}\\ &&+X_{acij\cdots}Y_{bdkl\cdots} + X_{bdij\cdots}Y_{ackl\cdots}\\ &&+X_{adij\cdots}Y_{bckl\cdots} + X_{bcij\cdots}Y_{adkl\cdots}\,.\end{aligned} \tag{A.13}$$

The indices which are not underlined are kept fixed in the original positions.

We have written the terms of order $t^{-1/2}$ in (A.7) and (A.8) because they play an important role in the computations of the anomalous term in (3.5). As discussed in Sec. III, the anomaly free requirement leads to the condition (3.6), and then all terms of order $t^{-1/2+m}(m \geq 0)$ do not contribute to final results. The formulas (A.7) to (A.12) are enough to solve the Wheeler-DeWitt equation to the second order in the strong coupling expansion.

APPENDIX B: THE FORMULAS (2.7) TO (2.10)

In this appendix, we will derive the equations (2.7) to (2.10) with the coefficients (2.11) and (2.12).

Substituting (2.4) into the left hand side of (2.7) and (2.8), we find

$$\Delta(x;t)\int d^3y\sqrt{h} = \lim_{x'\to x} K_{i'j'kl}(x',x;t)\frac{\sqrt{h(x)}}{4}$$
$$\times\left(h^{i'j'}(x)h^{kl}(x) - h^{i'k}(x)h^{j'l}(x) - h^{i'l}(x)h^{j'k}(x)\right) , \qquad (B.1)$$

$$\Delta(x;t)\int d^3y\sqrt{h}R$$
$$= \lim_{x'\to x}\left(K_{i'j'kl}(x',x;t)I^{i'j'kl}(x) + K_{i'j'kl;a'b'}(x',x;t)I^{i'j'kla'b'}(x)\right) , \qquad (B.2)$$

where

$$I^{i'j'kl} = \frac{\sqrt{h}}{2}\left\{\frac{1}{2}h^{i'j'}h^{kl}R - h^{i'k}h^{j'l}R\right.$$
$$\left. - h^{i'j'}R^{kl} - h^{kl}R^{i'j'} + 2h^{i'k}R^{j'l} + 2h^{i'l}R^{j'k}\right\} , \qquad (B.3)$$
$$I^{i'j'kla'b'} = \frac{\sqrt{h}}{2}\left\{h^{i'j'}h^{kl}h^{a'b'} - h^{i'j'}h^{kb'}h^{la'} - h^{i'b'}h^{j'a'}h^{kl}\right.$$
$$\left. - h^{i'k}h^{j'l}h^{a'b'} + 2h^{i'l}h^{j'b'}h^{ka'}\right\} .$$

Here, we have dropped surface terms. Using the results in Appendix A, we obtain

$$\Delta(x;t)\int d^3y\sqrt{h} = \frac{\sqrt{h(x)}}{(4\pi t)^{3/2}}\left\{-\frac{21}{8}\right\} + O(t^{-\frac{1}{2}}) , \qquad (B.4)$$

$$\Delta(x;t)\int d^3y\sqrt{h}R = \frac{\sqrt{h(x)}}{(4\pi t)^{3/2}}\left\{\frac{1}{t}\left(\frac{3}{2}\right) + \left(-\frac{11}{24} + \frac{3}{2}\,\xi\right)R(x)\right\} + O(t^{-\frac{1}{2}}) . \quad (B.5)$$

Computations of (2.9) and (2.10) are a little lengthy but straightforward. After some calculations, we have

$$\Delta(x;t)\int d^3y\sqrt{h}R^2$$
$$= \lim_{x'\to x}\left\{K_{i'j'kl}(x',x;t)J^{i'j'kl}(x) + K_{i'j'kl;a'}(x',x;t)J^{i'j'kla'}(x)\right.$$
$$+ K_{i'j'kl;a'b'}(x',x;t)J^{i'j'kla'b'}(x)$$
$$\left. + K_{i'j'kl;a'b'c'd'}(x',x;t)J^{i'j'kla'b'c'd'}(x)\right\} , \qquad (B.6)$$

$$\Delta(x;t)\int d^3y\sqrt{h}R_{ij}R^{ij}$$
$$= \lim_{x'\to x}\left\{K_{i'j'kl}(x',x;t)\overline{J}^{i'j'kl}(x) + K_{i'j'kl;a'}(x',x;t)\overline{J}^{i'j'kla'}(x)\right.$$
$$+ K_{i'j'kl;a'b'}(x',x;t)\overline{J}^{i'j'kla'b'}(x)$$
$$\left. + K_{i'j'kl;a'b'c'd'}(x',x;t)\overline{J}^{i'j'kla'b'c'd'}(x)\right\} , \qquad (B.7)$$

where

$$
\begin{aligned}
J^{i'j'kl} &= \sqrt{h}\Big\{(\tfrac{1}{4}h^{i'j'}h^{kl} - \tfrac{1}{2}h^{i'k}h^{j'l})R^2 \\
&\quad + (2h^{i'k}R^{j'l} + 2h^{j'k}R^{i'l} - h^{i'j'}R^{kl} - h^{kl}R^{i'j'})R \\
&\quad + 2R^{i'j'}R^{kl} + (2h^{ka'}h^{lb'} - 2h^{kl}h^{a'b'})R^{i'j'}{}_{;b'a'} \\
&\quad + (h^{kl}h^{i'j'}h^{a'b'} - 2h^{kl}h^{i'a'}h^{j'b'} - 2h^{i'k}h^{j'l}h^{a'b'} \\
&\quad - h^{ka'}h^{lb'}h^{i'j'} + 2h^{i'k}h^{la'}h^{j'b'} + 2h^{j'k}h^{lb'}h^{i'a'})R_{;b'a'}\Big\} \;,
\end{aligned}
$$

$$
\begin{aligned}
J^{i'j'kla'b'} &= \sqrt{h}\Big\{(h^{i'j'}h^{kl}h^{a'b'} + h^{i'k}h^{j'b'}h^{la'} + h^{b'i'}h^{j'l}h^{ka'} \\
&\quad - h^{i'k}h^{j'l}h^{a'b'} - h^{i'j'}h^{kb'}h^{la'} - h^{kl}h^{i'a'}h^{j'b'})R \\
&\quad + (2h^{ka'}h^{lb'} - 2h^{kl}h^{a'b'})R^{i'j'} + (2h^{i'a'}h^{j'b'} - 2h^{i'j'}h^{a'b'})R^{kl}\Big\} \;,
\end{aligned}
$$

$$
\begin{aligned}
J^{i'j'kla'b'c'd'} &= 2\sqrt{h}(h^{i'j'}h^{a'b'}h^{kl}h^{c'd'} - h^{i'a'}h^{j'b'}h^{kl}h^{c'd'} \\
&\quad - h^{i'j'}h^{a'b'}h^{kc'}h^{ld'} + h^{i'a'}h^{j'b'}h^{kc'}h^{ld'}) \;,
\end{aligned}
$$

$$
\begin{aligned}
\overline{J}^{i'j'kl} &= \sqrt{h}\Big\{(\tfrac{1}{4}h^{i'j'}h^{kl} - \tfrac{1}{2}h^{i'k}h^{j'l})R_{a'b'}R^{a'b'} + 2R^{i'k}R^{j'l} \\
&\quad - h^{kl}R^{i'}{}_{a'}R^{j'a'} - h^{i'j'}R^{k}{}_{a'}R^{la'} + 4h^{i'k}R^{j'}{}_{a'}R^{la'} \\
&\quad + (\tfrac{1}{2}h^{i'j'}h^{a'b'} - h^{i'a'}h^{j'b'})R^{kl}{}_{;b'a'} + (2h^{i'a'}h^{kb'} - 2h^{i'k}h^{a'b'})R^{j'l}{}_{;b'a'} \\
&\quad + (2h^{i'b'}h^{j'k} - h^{i'j'}h^{kb'})R^{la'}{}_{;b'a'} + (\tfrac{1}{2}h^{i'j'}h^{kl} - h^{i'k}h^{j'l})R^{a'b'}{}_{;b'a'} \\
&\quad + 2h^{i'k}h^{lb'}R^{j'a'}{}_{;b'a'} - 2h^{a'i'}h^{kl}R^{j'b'}{}_{;b'a'}\Big\} \;,
\end{aligned}
$$

$$
\begin{aligned}
\overline{J}^{i'j'kla'b'} &= \sqrt{h}\Big\{(2h^{i'b'}h^{ka'} - 2h^{a'b'}h^{i'k})R^{j'l} - 2h^{i'b'}h^{kl}R^{j'a'} \\
&\quad + 2h^{i'l}h^{ka'}R^{j'b'} + (2h^{i'b'}h^{j'k} - h^{i'j'}h^{kb'})R^{la'} \\
&\quad - h^{i'j'}h^{ka'}R^{lb'} + (h^{i'j'}h^{kl} - h^{i'k}h^{j'l})R^{a'b'}\Big\} \;,
\end{aligned}
$$

$$
\begin{aligned}
\overline{J}^{i'j'kla'b'c'd'} &= \tfrac{1}{2}\sqrt{h}\Big\{h^{a'b'}h^{i'k}h^{j'l}h^{c'd'} - h^{a'b'}h^{i'l}h^{j'd'}h^{kc'} - h^{a'b'}h^{i'd'}h^{j'l}h^{kc'} \\
&\quad + h^{a'b'}h^{i'd'}h^{j'c'}h^{kl} + h^{i'j'}h^{ka'}h^{lb'}h^{c'd'} - h^{i'j'}h^{la'}h^{b'd'}h^{kc'} \\
&\quad - h^{i'j'}h^{a'd'}h^{lb'}h^{kc'} + \tfrac{1}{2}h^{i'j'}h^{a'd'}h^{b'c'}h^{kl} + \tfrac{1}{2}h^{i'j'}h^{a'c'}h^{b'd'}h^{kl} \\
&\quad - 2h^{i'b'}h^{ka'}h^{j'l}h^{c'd'} + 2h^{i'b'}h^{la'}h^{j'd'}h^{kc'} + 2h^{i'b'}h^{a'd'}h^{j'l}h^{kc'} \\
&\quad - h^{i'b'}h^{a'd'}h^{j'c'}h^{kl} - h^{i'b'}h^{a'c'}h^{j'd'}h^{kl}\Big\} \;.
\end{aligned}
$$

$$
\text{(B.8)}
$$

Here we have not written $J^{i'j'kla'}$ and $\overline{J}^{i'j'kla'}$ because they do not contribute to the final results (2.9) and (2.10). Substitution of (A.7) to (A.12) into (B.6) and (B.7) leads to

$$
\begin{aligned}
&\Delta(x;t)\int d^3y\sqrt{h}R^2 \\
&= \frac{\sqrt{h(x)}}{(4\pi t)^{3/2}}\Big\{ \frac{1}{t}\big(-3R(x)\big) + \big(\frac{11}{8} - 3\,\xi\big)R^2(x) + \frac{31}{6}R_{ij}(x)R^{ij}(x) - R^{:i}{}_i(x) \Big\} \\
&\quad + O(t^{-\frac{1}{2}}) \;,
\end{aligned}
$$

$$
\text{(B.9)}
$$

$$\Delta(x;t) \int d^3y \sqrt{h} R_{ij} R^{ij}$$

$$= \frac{\sqrt{h(x)}}{(4\pi t)^{3/2}} \Big\{ \frac{1}{t^2}\Big(\frac{15}{4}\Big) + \frac{1}{t}\Big(-\frac{13}{8} + \frac{15}{4}\,\xi\Big) R(x) + \Big(-\frac{241}{480} - \frac{13}{8}\,\xi + \frac{15}{8}\,\xi^2\Big) R^2(x)$$

$$+ \frac{97}{20} R_{ij}(x) R^{ij}(x) + \Big(-\frac{11}{30} - \frac{25}{24}\,\xi\Big) R^{;i}{}_i(x) \Big\} + O(t^{-\frac{1}{2}}) \,. \qquad \text{(B.10)}$$

APPENDIX C: CONSISTENCY OF THE CONSTRAINTS

In this appendix we will compute the commutator (3.4). Let us first consider the commutator

$$\Big[\int d^3x \; \eta_1(x) \; \Delta_R(x) \;\;, \;\; \int d^3y \; \eta_2(y) \; \Delta_R(y) \Big] \,, \qquad \text{(C.1)}$$

where η_1 and η_2 are arbitrary scalar functions. We note that (C.1) is antisymmetric under the exchange of η_1 and η_2. The above commutator can be obtained from $[\int d^3x\eta_1(x)\Delta(x;t), \int d^3y\eta_2(x)\Delta(y;t')]$ by analytic continuation with respect to t and t', as discussed in Sec. II.

$$\Big[\int d^3x \; \eta_1(x) \; \Delta(x;t) \;\;, \;\; \int d^3y \; \eta_2(y) \; \Delta(y;t') \Big]$$

$$= \int d^3x d^3x' d^3y d^3y' \; \eta_1(x) \; \eta_2(y) \; K_{i'j'kl}(x',x;t)$$

$$\times \Big\{ \Big(\frac{\delta}{\delta h_{i'j'}(x')} K_{a'b'cd}(y',y;t') \Big) \frac{\delta}{\delta h_{kl}(x)} + \Big(\frac{\delta}{\delta h_{kl}(x)} K_{a'b'cd}(y',y;t') \Big) \frac{\delta}{\delta h_{i'j'}(x')}$$

$$+ \Big(\frac{\delta}{\delta h_{kl}(x)} \frac{\delta}{\delta h_{i'j'}(x')} K_{a'b'cd}(y',y;t') \Big) \Big\} \frac{\delta}{\delta h_{cd}(y)} \frac{\delta}{\delta h_{a'b'}(y')}$$

$$- \; (\; \eta_1 \; \leftrightarrow \; \eta_2 \;) \,. \qquad \text{(C.2)}$$

In our renormalization prescription, the cutoff t is removed by analytic continuation. This implies that we can take the naive limit $t \to 0$ if this limit produces no divergences in resulting calculations. Thus we may take the limit $t' \to 0$ in the first three terms on the right hand side of (C.2) and replace $K_{a'b'cd}(y',y;t')$ by $G_{a'b'cd}(y)\delta(y',y)$[10]. Then, the first three terms on the right hand side of (C.2) are reduced to

$$\int d^3x d^3y \; \eta_1(x) \; \eta_2(y)$$

$$\times \Big\{ K_{i'j'kl}(y,x;t) g^{i'j'}_{a'b'cd}(y) \frac{\delta}{\delta h_{kl}(x)} + \int d^3x' K_{i'j'kl}(x',x;t) g^{kl}_{a'b'cd}(y)\delta(y,x) \frac{\delta}{\delta h_{i'j'}(x')}$$

---

[10] In this paper, we will not rigorously justify taking the limit $t' \to 0$ at this stage, which might produce potential divergences due to the second functional derivatives $\frac{\delta}{\delta h_{cd}(y)} \frac{\delta}{\delta h_{a'b'}(y)}$ at the same point y.

$$
+ K_{i'j'kl}(y, x; t) g_{a'b'cd}^{i'j'kl}(y) \, \delta(y, x) \Big\} \frac{\delta}{\delta h_{cd}(y)} \frac{\delta}{\delta h_{a'b'}(y)} \ , \tag{C.3}
$$

where

$$
\frac{\delta}{\delta h_{i'j'}(x')} G_{a'b'cd}(y) \equiv g_{a'b'cd}^{i'j'}(y) \, \delta(y, x') \ ,
$$

$$
\frac{\delta}{\delta h_{kl}(x)} \frac{\delta}{\delta h_{i'j'}(x')} G_{a'b'cd}(y) \equiv g_{a'b'cd}^{i'j'kl}(y) \, \delta(y, x') \, \delta(y, x) \ . \tag{C.4}
$$

Taking the limit $t \to 0$ in the first two terms of (C.3) but keeping t nonzero in the last term of (C.3), we have

$$
\int d^3 x \; \eta_1(x) \; \eta_2(x) \left\{ \; G_{i'j'kl}(x) \; g_{a'b'cd}^{i'j'}(x) \frac{\delta}{\delta h_{kl}(x)} \right.
$$

$$
\left. + \; G_{i'j'kl}(x) \; g_{a'b'cd}^{kl}(x) \frac{\delta}{\delta h_{i'j'}(x)} + K_{i'j'kl}(x, x; t) \; g_{a'b'cd}^{i'j'kl}(x) \right\} \frac{\delta}{\delta h_{cd}(x)} \frac{\delta}{\delta h_{a'b'}(x)} \ . \tag{C.5}
$$

This is clearly symmetric under the exchange of η_1 and η_2. Since only antisymmetric terms under the exchange of η_1 and η_2 survive in the commutator (C.1), we conclude that

$$
\left[\; \int d^3 x \; \eta_1(x) \; \Delta_R(x) \; , \; \int d^3 y \; \eta_2(y) \; \Delta_R(y) \; \right] \; = \; 0 \ . \tag{C.6}
$$

Therefore, nontrivial contributions to the commutator (3.4) come from

$$
\left[\; -16\pi G \int d^3 x \; \eta_1(x) \; \Delta_R(x) \; , \; \frac{1}{16\pi G} \int d^3 y \; \eta_2(y) \; \sqrt{h(y)} \; (R(y) + 2\Lambda) \; \right]
$$

$$
- \; (\; \eta_1 \; \leftrightarrow \; \eta_2 \;) \ . \tag{C.7}
$$

As before, the above commutator can be obtained, by analytic continuation with respect to t, from

$$
\left[\; -16\pi G \int d^3 x \; \eta_1(x) \; \Delta(x; t) \; , \; \frac{1}{16\pi G} \int d^3 y \; \eta_2(y) \; \sqrt{h(y)} \; (R(y) + 2\Lambda) \; \right]
$$

$$
- \; (\; \eta_1 \; \leftrightarrow \; \eta_2 \;)
$$

$$
= \int d^3 x d^3 x' d^3 y \; \eta_1(x) \; \eta_2(y) \; K_{i'j'kl}(x', x; t)
$$

$$
\times \left\{ \; - \left(\frac{\delta}{\delta h_{i'j'}(x')} \sqrt{h(y)} \; (R(y) + 2\Lambda) \right) \frac{\delta}{\delta h_{kl}(x)} - \left(\frac{\delta}{\delta h_{kl}(x)} \sqrt{h(y)} \; (R(y) + 2\Lambda) \right) \frac{\delta}{\delta h_{i'j'}(x')} \right.
$$

$$
\left. - \left(\frac{\delta}{\delta h_{kl}(x)} \frac{\delta}{\delta h_{i'j'}(x')} \sqrt{h(y)} \; (R(y) + 2\Lambda) \right) \right\}
$$

$$
- \; (\; \eta_1 \; \leftrightarrow \; \eta_2 \;) \ . \tag{C.8}
$$

We can take the limit $t \to 0$ without any trouble in the first two terms on the right hand side of (C.8). It is then easy to see that these two terms give the first term

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on the right hand side of (3.4). We cannot, however, take the limit $t \to 0$ in the third term on the right hand side of (C.8). The third term would give an anomalous contribution to the commutator. It is not difficult to show that

$$
\int d^3x d^3x' d^3y \; \eta_1(x) \; \eta_2(y) \; K_{i'j'kl}(x', x; t) \left(-\frac{\delta}{\delta h_{kl}(x)} \frac{\delta}{\delta h_{i'j'}(x')} \right) \sqrt{h(y)} \; (R(y) + 2\Lambda)
$$
$$
- \; (\; \eta_1 \; \leftrightarrow \; \eta_2 \;)
$$
$$
= -\frac{\frac{1}{24} + \xi}{(4\pi)^{3/2} t^{1/2}} \int d^3x \sqrt{h(x)} \left(\eta_1(x)(\nabla_i \, \eta_2(x)) - (\nabla_i \, \eta_1(x))\eta_2(x) \right) \nabla^i R(x) + O(t^{\frac{1}{2}}) \; .
$$

$$(C.9)$$

Therefore, our renormalization prescription tells us that

$$
\left[\; -16\pi G \int d^3x \; \eta_1(x) \; \Delta_R(x) \;\; , \;\; \frac{1}{16\pi G} \int d^3y \; \eta_2(y) \; \sqrt{h(y)} \; (R(y) + 2\Lambda) \; \right]
$$
$$
- \; (\; \eta_1 \; \leftrightarrow \; \eta_2 \;)
$$
$$
= \int d^3x \left(\eta_1(x)(\nabla_i \, \eta_2(x)) - (\nabla_i \, \eta_1(x))\eta_2(x) \right)
$$
$$
\times \left(\; i \; \mathcal{H}^i(x) - \frac{\frac{1}{24} + \xi}{(4\pi)^{3/2}} \phi^{(1)}(0) \sqrt{h(x)} \; \nabla^i R(x) \; \right) \; .
$$

$$(C.10)$$

Combining (C.6) with (C.10), we get the result (3.4) with the anomalous term (3.5).

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